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Wasserstein metric convergence method for Fokker–Planck equations with point controls

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ABSTRACT

Monge–Kantorovich mass transfer theory is employed to obtain an existence and uniqueness result for solutions to Fokker–Planck Equations with time dependent point control. Existence for an approximate problem is established together with a convergence analysis in the *Wasserstein distance* through equivalence with weak-$\star$ convergence.

1. Introduction

A new result for linear diffusion equations with point controls is presented. This result is closely related to similar work on more general nonlinear diffusion equations [1]. The focus of this paper is illustrating how recent variational principles based on *Wasserstein metric*, used to solve homogeneous diffusion equations, can be extended to solve non-homogeneous equations. In particular, we study here an initial value problem from control theory.

We consider a Fokker–Planck Equation (FPE) in one dimension with a time dependent point control of this form:

$$\begin{cases} \rho_t + (A\,\rho)_x - \dfrac{B}{2}\,\rho_{xx} = f + v(t)\,\delta(x-a) & \text{in } \Omega \times [0,T] \\ A(x)\,\rho(x,t) - \dfrac{B}{2}\,\rho_x(x,t) = 0 & \text{on } \partial\Omega \times [0,T] \\ \rho(x,0) = \rho_0(x) & \text{in } \Omega \end{cases} \tag{1.1}$$

where B is the positive constant coefficient of diffusion, $f(x,t)$, $v(t)$ and $A(x)$ satisfy the following conditions:

$$\begin{cases} A \in C^1(\Omega), \quad A(x) \geq 0 \quad \text{for all } x \in \Omega \\ f(\cdot,t) \in L^1(\Omega) \quad \text{for all } t \in [0,T], \qquad f(x,t) \geq 0 \quad \text{in } \Omega \times [0,T] \\ v(t) \geq 0. \end{cases} \tag{1.2}$$

We prove existence and uniqueness of a weak solution via a Wasserstein convergence method. Namely we will prove existence and uniqueness for the following approximate problem, where $\mu_\epsilon(x)$ is a smooth approximation of the Dirac

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Delta function:

$$\begin{cases} \rho_{\varepsilon,t} + (A\,\rho_\varepsilon)_x - \dfrac{B}{2}\,\rho_{\varepsilon,xx} = f + v(t)\,\mu_\varepsilon(x) & \text{in } \Omega \times [0,T] \\ A(x)\,\rho_\varepsilon(x,t) - \dfrac{B}{2}\,\rho_{\varepsilon,x}(x,t) = 0 & \text{on } \partial\Omega \times [0,T] \\ \rho_\varepsilon(x,0) = \rho_0(x) & \text{in } \Omega. \end{cases} \tag{1.3}$$

We remark that the introduction of this approximate problem is unfortunately a necessary step in the proof. More details about this are given in the next section. We then use the transposition method and the adjoint to the FPE to prove the main convergence result in the Wasserstein metric. This requires a major change of the usual Wasserstein metric variational formulation. Due to the presence of a forcing term on the right-hand side the masses are no longer conserved and therefore the usual comparison between measures does not work. In order to analyze this we follow the idea of Kinderlehrer and Walkington in [2] and modify the variational principle accordingly.

2. Existence of solution for the approximate FPE problem

The results obtained by Jordan, Kinderlehrer and Otto in [3] for a homogeneous FPE and Kinderlehrer and Walkington in [2] for a non-homogeneous one, can be manipulated to show the existence of a unique minimizer for the scheme:

$$\begin{cases} \text{Determine } \rho^{(k)}_{\tau,\epsilon} \text{ that minimizes} \\ \dfrac{1}{2}\mathrm{d}\left(\rho^{(k-1)}_{\tau,\varepsilon} + \tau\,(f + v\mu_\varepsilon), \rho\right)^2 + \tau\,F(\rho) \\ \text{over all } \rho \in K_k, \quad k = 1, 2, \ldots \end{cases} \tag{2.1}$$

where d is the 2-Wasserstein distance, $F(\rho) = \int_\Omega \left(E(x)\,\rho + \frac{B}{2}\,\rho\,\log\,\rho\right)\mathrm{d}x$, $E(x)$ is a function such that $E'(x) = -A(x)$ and $E(x) \geq 0$ and the K_k's are the sets of admissible densities, i.e.:

$$K_k := \left\{\rho \geq 0 \text{ meas., s.t. } \int_\Omega \rho(x)\mathrm{d}x = \int_\Omega \rho^{(k-1)}(x) + \tau\,(f + v\,\mu_\varepsilon)\mathrm{d}x\right\}.$$

We denote $\rho^{(0)} = \rho_0$, and let $\rho^{(k)}_{\tau,\epsilon}$ be the minimizer of (2.1) and define $\rho_{\tau,\varepsilon}$ to be the interpolated function $\rho_{\tau,\varepsilon}(t) = \rho^{(k)}_{\tau,\varepsilon}$ for $t \in [k\tau, (k+1)\tau)$ and $k \in N \cup \{0\}$.

It is clear why the introduction of the approximate problem (1.3) and the related variational principle (2.1) are necessary [4]. In the case of a Dirac delta function $F(\rho)$ the energy functional or entropy of the system is not defined.

We then have the following proposition:

Proposition 1. *Given $\rho^{(k-1)}_{\tau,\varepsilon} \in K_{(k-1)}$, there exists a unique solution of the scheme (2.1).*

Proof. It can be shown, just as in [3], that the functional F is well defined as a functional on K_k and that $\frac{1}{2}\,\mathrm{d}(\rho^{(k-1)}_{\tau,\epsilon} + \tau\,(f + v\,\mu_\epsilon)\,,\ \rho)^2 + \tau\,F(\rho)$ is bounded below. Then, letting ρ_ν be a minimizing sequence and using a Dunford–Pettis criteria we have (at least for a subsequence) weak convergence in $L^1(\Omega)$. For details see [3]. □

The main result for the approximation problem, i.e. the convergence of the solution of (2.1) to the solution of (1.3), we state in the following theorem. Dependence on ε is not specifically stated in our notation:

Theorem 1. *Let $\rho^{(0)} \in K_0$ satisfy $F(\rho^{(0)}) < \infty$, and for given $\tau > 0$, let $\{\rho^{(k)}_\tau\}_{k\in N}$ be the solution of (2.1). Define the interpolation $\rho_\tau : (0,\infty) \times \Omega \to [0,\infty)$ by*

$$\rho_\tau(t) = \rho^{(k)}_\tau \quad \text{for } t \in [k\tau, (k+1)\tau) \quad \text{and} \quad k \in N \cup \{0\}.$$

Then as $\tau \downarrow 0$,

$$\rho_\tau(t) \rightharpoonup \rho(t) \quad \text{weakly in } L^1(\Omega) \text{ for all } t \in (0,\infty), \tag{2.2}$$

where $\rho \in C^\infty((0,\infty) \times \Omega)$ is the unique solution of:

$$\rho_t + (A\,\rho)_x - \frac{B}{2}\,\rho_{xx} = f + v\,\mu \tag{2.3}$$

with initial condition:

$$\rho(t) \to \rho^0 \quad \text{strongly in } L^1(\Omega) \text{ for } t \downarrow 0, \tag{2.4}$$

boundary condition:

$$A(x)\,\rho(x,t) - \frac{B}{2}\,\rho_x(x,t) = 0 \quad \text{on } \partial\Omega \times [0,T] \tag{2.5}$$

and

$$M(\rho), F(\rho) \in L^{\infty}((0,T)) \quad \text{for all } T < \infty. \tag{2.6}$$

Proof. The proof of this theorem is similar to that found in [3] and is comprised of 4 parts. First we show that the scheme (2.1) delivers the proper weak equations, then we prove that (2.1) gives the natural boundary conditions on Σ. In the third part we will prove some inequalities that are needed later. In the last part we reorganize the results. □

2.1. Weak equations

Weak equations will be derived using a method used by Otto in [5] called Variation of Domain. Define $y = \psi(x,\varepsilon) = \psi_\varepsilon(x)$ by $\frac{\mathrm{d}y}{\mathrm{d}\varepsilon} = \xi(y)$ and $y|_{\varepsilon=0} = \psi_0(x) = x$ and ρ_ε "push forward" of ρ by:

$$\int_\Omega \xi(x)\,\rho_\varepsilon(x)\mathrm{d}x = \int_\Omega \xi(\psi_\varepsilon(x))\,\rho(x)\mathrm{d}x = \int_\Omega \xi(y)\,\frac{\rho(\psi_\varepsilon^{-1}(y))}{\frac{\mathrm{d}\psi_\varepsilon}{\mathrm{d}x}(\psi_\varepsilon^{-1}(y))}\mathrm{d}y$$
$$\Rightarrow \rho_\varepsilon(y) = \frac{\rho(\psi_\varepsilon^{-1}(y))}{\frac{\mathrm{d}\psi_\varepsilon}{\mathrm{d}x}(\psi_\varepsilon^{-1}(y))}.$$

Recall $F(\rho) = \int_\Omega (E\,\rho + \frac{B}{2}\,\rho\,\log\,\rho)\mathrm{d}x$ then we need $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}F(\rho_\varepsilon)|_{\varepsilon=0}$. Consider the second term of the integral, discarding constants, then we have:

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_\Omega \rho_\varepsilon(x)\log(\rho_\varepsilon(x))\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_\Omega \rho\log(\rho_\varepsilon(\psi_\varepsilon(x)))\mathrm{d}x = \int_\Omega \rho\,\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\log(\rho_\varepsilon(\psi_\varepsilon(x)))\mathrm{d}x$$

assuming we can interchange the derivation and the integration by, for example, approximating the log function:

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}[\log(\rho_\varepsilon(\psi_\varepsilon(x)))] = \frac{\frac{\mathrm{d}}{\mathrm{d}\varepsilon}(\rho_\varepsilon(\psi_\varepsilon(x)))}{\rho_\varepsilon(\psi_\varepsilon(x))} = -\frac{\xi'(\psi_\varepsilon(x))\,\rho_\varepsilon(\psi_\varepsilon(x))}{\rho_\varepsilon(\psi_\varepsilon(x))}$$

and after simplifying and setting $\varepsilon = 0$ we get:

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}[\log(\rho_\varepsilon(\psi_\varepsilon(x)))]\bigg|_{\varepsilon=0} = -\,\xi'(\psi_0(x)) = -\,\xi'(x).$$

Reassuming:

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_\Omega \frac{B}{2}\,\rho\,\log\,\rho\mathrm{d}x\bigg|_{\varepsilon=0} = -\,\frac{B}{2}\int_\Omega \rho(x)\,\xi'(x)\mathrm{d}x.$$

For the first term we have:

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0}\int_\Omega E(x)\,\rho_\varepsilon(x)\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0}\int_\Omega E(\psi_\varepsilon(x))\,\rho(x)\mathrm{d}x = \int_\Omega \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} E(\psi_\varepsilon(x))\,\rho(x)\mathrm{d}x$$
$$= \int_\Omega (-A(x)\,\rho(x)\,\xi(x))\,\mathrm{d}x.$$

Let now $\rho + \tau(f + v\,\mu) = \tilde{\rho}$ for notational convenience and define $p_\varepsilon(x,y)$ by:

$$\int_\Omega\int_\Omega \xi(x,y)\mathrm{d}p_\varepsilon(x,y) = \int_\Omega\int_\Omega \xi(x,\psi_\varepsilon(y))\mathrm{d}p(x,y)$$

then:

$$\frac{1}{2\varepsilon}\left[\mathrm{d}(\rho_\varepsilon,\tilde{\rho})^2 - \mathrm{d}(\rho,\tilde{\rho})^2\right] \le \frac{1}{2\varepsilon}\int_\Omega\int_\Omega \left((x-\psi_\varepsilon(y))^2 - (x-y)^2\right)\mathrm{d}p(x,y)$$

and taking the $\limsup_{\varepsilon\to 0}$ and using the definition of the Wasserstein distance:

$$0 \le -\int_\Omega\int_\Omega (x-y)\,\xi(y)\mathrm{d}p(x,y).$$

Combining all of these results:

$$0 \le -\int_\Omega\int_\Omega (x-y)\,\xi(y)\mathrm{d}p(x,y) - \tau\int_\Omega \left(A(x)\,\rho(x)\,\xi(x) + \frac{B}{2}\,\rho(x)\,\xi'(x)\right)\mathrm{d}x$$

it is clearly possible to change ξ with $-\xi$ and therefore obtain:

$$0 = \int_\Omega \int_\Omega (x-y)\,\xi(y)\mathrm{d}p(x,y) \;+\; \tau \int_\Omega \left(A(x)\,\rho(x)\,\xi(x) + \frac{B}{2}\,\rho(x)\,\xi'(x) \right) \mathrm{d}x. \tag{2.7}$$

Let now $\xi(y) = \zeta'(y)$ then by a simple Taylor's expansion argument $\zeta(x) - \zeta(y) = \zeta'(y)(x-y) + \text{h.o.t.}$ where

$$\text{h.o.t.} = \frac{1}{2}\zeta''()(x-y)^2 = O(\|\zeta''\|_\infty)(x-y)^2$$

therefore:

$$\begin{aligned}\int_\Omega \int_\Omega (x-y)\,\zeta'(y)\mathrm{d}p(x,y) &= \int_\Omega \int_\Omega (\zeta(x) - \zeta(y))\mathrm{d}p(x,y) + O\left(\|\zeta''\|_\infty\right) \int_\Omega \int_\Omega (x-y)^2 \mathrm{d}p(x,y)\\ &= \int_\Omega \int_\Omega (\zeta(x) - \zeta(y))\mathrm{d}p(x,y) + O\left(\|\zeta''\|_\infty \mathrm{d}(\rho,\tilde{\rho})^2\right)\\ &= \int_\Omega (\rho(x) - \tilde{\rho}(x))\,\xi(x)\mathrm{d}x + O\left(\|\zeta''\|_\infty \mathrm{d}(\rho,\tilde{\rho})^2\right)\end{aligned}$$

where we use again the definition of the Wasserstein distance in the last step. Substituting in (2.7), integrating by parts, and considering the absolute value we get:

$$\left| \int_\Omega (\rho - \tilde{\rho})\,\xi \mathrm{d}x + \tau \int_\Omega \left(A\,\rho\,\xi + \frac{B}{2}\,\rho\,\xi' \right) \mathrm{d}x \right| = \left| \int_\Omega \left[\rho - \tilde{\rho} + \tau \left(A\,\rho - \frac{B}{2}\,\rho_x \right)_x \right] \xi \mathrm{d}x \right| \;\le\; O\left(\|\zeta''\|_\infty \mathrm{d}(\rho,\tilde{\rho})^2 \right).$$

Recalling that $\frac{\rho - \tilde{\rho}}{\tau} = f + v\mu$ and letting $\tau \downarrow 0$ gives the desired weak equation, see [2] for details:

$$\rho_t + (A\,\rho)_x - \frac{B}{2}\,\rho_{xx} = f + v\,\mu.$$

2.2. Boundary condition

To check that (2.1) gives us the natural boundary conditions (2.5) we use a standard variation argument. Let $\rho_\varepsilon = \rho + \varepsilon\,\xi$, let $\xi = \zeta'$ s.t. $\zeta(0) = \zeta(1)$, i.e. choose ξ with average 0, then:

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} F(\rho_\varepsilon)\bigg|_{\varepsilon=0} = \int_\Omega \left(E + \frac{B}{2}\,(1 + \log\,\rho) \right) \xi(x)\mathrm{d}x$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathrm{d}(\rho_\varepsilon,\,\tilde{\rho})^2\bigg|_{\varepsilon=0} = -2 \int_\Omega \left(\int_x^1 (s - \phi(s))\mathrm{d}s \right) \xi(x)\mathrm{d}x,$$

using the representation of the Wasserstein distance in one dimension, given by the distribution functions of $\tilde{\rho}$ and ρ (say V and U resp.), i.e.

$$\mathrm{d}(\rho,\,\tilde{\rho})^2 = \int_\Omega (x - \phi(x))^2 \rho(x)\mathrm{d}x = \int_\Omega \left(x - V^{-1} \circ U(x) \right)^2 \rho(x)\mathrm{d}x.$$

We also get $\rho(x) = \phi'(x)\,\tilde{\rho}(\phi(x))$. Thus when (2.1) holds:

$$-\int_\Omega \left(\int_x^1 (s - \phi(s))\mathrm{d}s \right) \xi(x)\mathrm{d}x + \tau \int_\Omega \left(E(x) + \frac{B}{2}\,(1 + \log\,\rho(x)) \right) \xi(x)\mathrm{d}x = 0$$

whenever ξ has average 0. This implies that:

$$\frac{\partial}{\partial x} \left(-\int_x^1 (s - \phi(s))\mathrm{d}s + \tau \left(E(x) + \frac{B}{2}\,(1 + \log\,\rho(x)) \right) \right) = 0$$

or

$$(x - \phi(x)) + \tau \left(-A(x) + \frac{B}{2}\,\frac{\rho_x(x)}{\rho(x)} \right) = 0.$$

In particular for $x = 0$ and $x = 1$ when $x = \phi(x)$ we have:

$$A(x)\,\rho(x) - \frac{B}{2}\,\rho_x(x) = 0.$$

2.3. Inequalities

Following the proof given in [3] we will need to prove the following inequalities in order to establish (2.2) in the limit when $\tau \downarrow 0$. More specifically that for any $T < \infty$, there exists a constant $C < \infty$ such that for all $N \in \mathbf{N}$ and all $\tau \in [0, 1]$ with $N\tau < T$, there holds

$$M(\rho_\tau^{(N)}) \le C \tag{2.8}$$

$$\int_{\mathfrak{R}} \max\left\{\rho_\tau^{(N)} \log \rho_\tau^{(N)}, 0\right\} \mathrm{d}x \le C \tag{2.9}$$

$$\int_{\mathfrak{R}} E\, \rho_\tau^{(k)} \mathrm{d}x \le C \tag{2.10}$$

$$\sum_{k=1}^{N} d\left(\rho_\tau^{(k-1)} + \tau(f + v\mu), \rho_\tau^{(k)}\right)^2 \le C\tau. \tag{2.11}$$

To prove (2.8) note that $\Omega = (0, 1)$ is bounded and $N\tau = T$, therefore:

$$M(\rho_\tau^{(N)}) = \int_\Omega x^2\, \rho_\tau^{(N)}(x)\mathrm{d}x \le \int_0^1 \rho_\tau^{(N)}\mathrm{d}x \le 1 + N\int_\Omega \tau(f + v\,\mu_\varepsilon)\mathrm{d}x \le C.$$

To show (2.9) and (2.10) we refer to [3], the proofs being similar. In order to show the last inequality (2.11) we need to take into consideration the non-homogeneity of the FPE we consider. The linear term in the integral in (2.1) clearly presents no problem, while for the nonlinear one we will use the convexity of $L(\rho) = \int \rho\, \log \rho \mathrm{d}x$.

Obviously $\rho_\tau^{(k-1)} + \tau(f + v\mu)$ is an admissible density in the search for the minimum in (2.1), denoting $(f + v\mu) = g$ we have as follows:

$$\frac{1}{2\tau}\mathrm{d}\left(\rho_\tau^{(k-1)} + \tau\, g, \rho_\tau^{(k)}\right)^2 + \int_\Omega \left(E\,\rho_\tau^{(k)} + \frac{B}{2}\,\rho_\tau^{(k)}\, \log \rho_\tau^{(k)}\right)\mathrm{d}x$$
$$\le \int_\Omega \left(E\,\left(\rho_\tau^{(k-1)} + \tau\, g\right) + \frac{B}{2}\,\left(\rho_\tau^{(k-1)} + \tau\, g\right)\log\left(\rho_\tau^{(k-1)} + \tau\, g\right)\right)\mathrm{d}x.$$

From the convexity property we get:

$$L\left(\rho_\tau^{(k-1)} + \tau\, g\right) \le \frac{1}{2}L\left(2\rho_\tau^{(k-1)}\right) + \frac{1}{2}L\left(2\tau\, g\right) \le L\left(\rho_\tau^{(k-1)}\right) + C.$$

Summing over K and noting that we get telescoping sums :

$$\frac{1}{2\tau}\sum_{k=1}^{N}\mathrm{d}\left(\rho_\tau^{(k-1)} + \tau\, g, \rho_\tau^{(k)}\right)^2 \le \frac{B}{2}\left(\rho_\tau^{(0)} - \rho_\tau^{(N)} + C\,N\right) + \int_\Omega E\left(\rho_\tau^{(0)} - \rho_\tau^{(N)}\right)\mathrm{d}x$$
$$+N\,\tau\int_\Omega E\,g\mathrm{d}x \le C\,N + C + T\int_\Omega E\,g\mathrm{d}x \le C\,N,$$

where in the last line we used (2.8) and (2.10) and the boundedness of the function g.

2.4. Proof of main theorem

Conclusion of proof of Theorem 1. We resume the dependence on ε. Using a Dunford–Pettis like criteria we have, owing to estimates (2.8) and (2.9) that there exists a measurable $\rho_\varepsilon(x, t)$ such that, after extraction of a subsequence we have:

$$\rho_{\tau,\varepsilon} \rightharpoonup \rho_\varepsilon \quad \text{weakly in } L^1((0, T) \times (\Omega)) \text{ for all } T < \infty. \tag{2.12}$$

The estimates (2.8)–(2.10) guarantee that $\rho_\varepsilon(t) \in K$ for a.e. $t \in (0, \infty)$ and that we also have (2.6). We can furthermore extend the convergence to (2.2) and the regularity of the solution $\rho_\varepsilon(x, t)$, just as in [3]. Finally, uniqueness follows from the linearity of the equation. □

3. Convergence result

The main result of the paper is the existence and uniqueness of a solution of (1.1). We have the following:

Theorem 2. *Let $\rho^{(0)} \in K_0$, then there exists ρ_ε solution to (1.3) such that $\rho_\varepsilon \rightharpoonup \rho$ weakly $*$ (or equivalently $\mathrm{d}(\rho_\varepsilon, \rho) \to 0$) where ρ is the unique solution of (1.1).*

Proof. From Theorem 1 we have existence of such a ρ_ε solution of (1.3). Let us now define an adjoint to the FPE Eq. (1.1) as:

$$\begin{cases} -\phi_t - A\,\phi_x - \frac{B}{2}\,\phi_{xx} = \psi & \text{in } \Omega \times [0,T] \\ \phi_x = 0 & \text{on } \partial\Omega \times [0,T] \\ \phi(x,T) = 0 & \text{in } \Omega \end{cases} \tag{3.1}$$

with ψ a test function. Now define ρ and ϕ, the solutions of (1.1) and (3.1) respectively, which exist, (see for example [6]). Then by using integration by parts in the following manner we can rewrite:

$$\begin{aligned} \int_0^T \int_\Omega \psi\,\rho \mathrm{d}x\mathrm{d}t &= \int_0^T \int_\Omega \left(-\phi_t - A\,\phi_x - \frac{B}{2}\,\phi_{xx}\right)\rho \mathrm{d}x\mathrm{d}t \\ &= \int_0^T \int_\Omega \phi\left(\rho_t + (A\,\rho)_x - \frac{B}{2}\,\rho_{xx}\right)\mathrm{d}x\mathrm{d}t - \int_0^T \phi\left(A\,\rho - \frac{B}{2}\,\rho_x\right)\Big|_{\partial\Omega}\mathrm{d}t \\ &\quad - \int_\Omega \rho\,\phi\Big|_0^T \mathrm{d}x - \frac{B}{2}\int_0^T (\rho\,\phi_x)\Big|_{\partial\Omega}\mathrm{d}t \\ &= \int_0^T \int_\Omega (f + v\,\delta(x-a))\,\phi \mathrm{d}x\mathrm{d}t + \int_\Omega \rho_0(x)\,\phi(x,0)\mathrm{d}x. \end{aligned}$$

We can similarly rewrite for ρ_ε:

$$\int_0^T \int_\Omega \psi\,\rho_\varepsilon \mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega (f + v\,\mu_\varepsilon)\,\phi \mathrm{d}x\mathrm{d}t + \int_\Omega \rho_0(x)\,\phi(x,0)\mathrm{d}x.$$

Then we have:

$$\int_0^T \int_\Omega \psi(\rho - \rho_\varepsilon)\mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega \phi\,v\,(\delta - \mu_\varepsilon)\mathrm{d}x\mathrm{d}t \longrightarrow 0,$$

as ϕ is continuous in x and $\mu_\varepsilon \rightharpoonup \delta$. We have thus proved weak-$\star$ convergence, which implies Wasserstein distance convergence, of the solution of (1.3) to (1.1). Uniqueness, once again, follows easily from the linearity of the equations involved. □

4. Application

The problem of calibrating measurement instruments is often solved using mathematical formulations based on control theory. The instrument controls and instrument outputs are modeled as control and state variables respectively. Recent efforts to calibrate one specific instrument, a Matrix-assisted laser desorption/ionization time-of-flight mass spectrometry Time of Flight (MALDI/TOF) instrument, have led to the need to minimize functions that are noise-ridden [7,8]. The calculation of noise-adjusted gradients of these functions can be accomplished [9] but only if there is an understanding of the noise and how it affects the calibration function. Attempts to model this noise, [10,9] have led to models that could yield this information about the way noise infects data produced by MALDI/TOF instruments. However making use of this information would require the solution and control of Fokker–Plank equations using non-traditional discretization methods.

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